Generalization of the Poisson kernel to the superconducting random-matrix ensembles

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We calculate the distribution of the scattering matrix at the Fermi level for chaotic normal-superconducting systems for the case of arbitrary coupling of the scattering region to the scattering channels. The derivation is based on the assumption of uniformly distributed scattering matrices at ideal coupling, which holds in the absence of a gap in the quasiparticle excitation spectrum. The resulting distribution generalizes the Poisson kernel to the nonstandard symmetry classes introduced by Altland and Zirnbauer. We show that unlike the Poisson kernel, our result cannot be obtained by combining the maximum entropy principle with the analyticity-ergodicity constraint. As a simple application, we calculate the distribution of the conductance for a single-channel chaotic Andreev quantum dot in a magnetic field.

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I. INTRODUCTION

Statistical aspects of electronic transport through chaotic cavities (quantum dots) can be efficiently described using a random matrix model for the $N \times N$ unitary scattering matrix S of the system (see Ref. 1 for a review). For sufficiently low temperatures and voltages, transport properties can be expressed by the scattering matrix at the Fermi level. Besides unitarity, a crucial role is played in the random matrix models by the additional constraints satisfied by S, defining the so-called symmetry classes.

In the absence of superconductivity, following Dyson,[2] three symmetry classes are distinguished, depending on the presence or absence of time-reversal and spin-rotation symmetry. In this classification scheme, the cases are labeled by the index β, and the additional constraints on S are as follows. In the presence of time-reversal, as well as spin-rotation symmetry ($\beta = 1$), S is symmetric, $S = S^T$. In the absence of time-reversal symmetry ($\beta = 2$), the only requirement is the unitarity of S. In the presence of time-reversal symmetry, but without spin-rotation invariance ($\beta = 4$), S is self-dual, $S = S^R$. (The dual of a matrix A is defined by $A^R = \tau A^T \tau^T$, with $\tau = i\sigma_2$, where σ_j denotes the j-th Pauli matrix in spin space.) It has been shown[3] that for ideal coupling of the scattering channels to the cavity, i.e., in the absence of direct reflection from the cavity openings, the distribution of S is uniform. Uniformity is understood with respect to the invariant measure in the unitary group subject to the constraints imposed by the symmetries under spin-rotation and time-reversal. From the uniform distribution at ideal coupling, it follows[3] that at arbitrary coupling the probability density of S is given by the Poisson kernel[3,4,5,6,7,8]

$$P_\beta(S) \propto |\det(1 - r^\dagger S)|^{-(\beta N + 2 - \beta)} \qquad (1)$$

where r is the matrix describing the direct reflections from the openings.

Dyson's classification scheme becomes insufficient in the presence of superconductivity.[9,10] In normal-superconducting hybrid systems, the scattering ma-

trix acquires an electron-hole structure,[11,12,13] and it satisfies[10] a constraint at the Fermi level, $S = \Sigma_1 S^* \Sigma_1$, expressing the electron-hole symmetry. (Σ_j denotes the j-th Pauli matrix in electron hole space.) Altland and Zirnbauer[10] showed that depending on the symmetries under time-reversal and spin-rotation, these systems fall into four new symmetry classes, which they labeled following Cartan's notation of the corresponding symmetric spaces. Systems where both symmetries are broken, belong to class D. If only spin-rotation invariance is broken, class DIII is realized. If only time-reversal symmetry is broken, the system belongs to class C, and finally, if all symmetries are present, the system belongs to class CI. The requirements for S following from time-reversal and spin-rotation symmetry are the same as in the absence of superconductivity. Assuming gapless quasiparticle excitations, Altland and Zirnbauer introduced a random scattering matrix model for transport in chaotic normal-superconducting systems, by adopting a uniform distribution for the scattering matrix. This is appropriate for the case when the coupling of the cavity to the transport channels is ideal. The analogue of the Poisson kernel, i.e., the distribution of S, for the case of arbitrary coupling, to the best of our knowledge, has not been presented yet. In this study, we aim at providing this result. We believe that the knowledge of this distribution is desirable, as it can serve as a starting point to extend results that are based on the Poisson kernel $P_\beta(S)$, from Dyson's standard symmetry classes to the classes of Altland and Zirnbauer. As a particular example we mention the study of dephasing in the framework of Büttiker's dephasing lead model.[14,15,16,17] In this model, in order to account for dephasing mechanisms that occur uniformly in the quantum dot, the knowledge of the distribution of the scattering matrix for nonideal coupling is essential.[18]

The paper is organized as follows. In the next Section, we relate the attributes of the scattering matrix to those of normal-superconducting quantum dots, and we briefly discuss the conditions at which a random scattering matrix description for the transport in such systems is adequate. In Sec. III, we detail the properties of the

manifolds in the space of $N \times N$ matrices defined by the constraints on the scattering matrix corresponding to the symmetry classes of Altland and Zirnbauer. In Sec. IV, we present the calculation of the distribution $P(S)$ of the scattering matrix, based on the assumption that S is uniformly distributed in the ideal coupling case. We illustrate the use of our result in Sec. V on a simple but physically realistic example, a single mode normal-superconducting quantum dot in magnetic field. We conclude in Sec. VI by contrasting $P(S)$ and the Poisson kernel $P_\beta(S)$ regarding the applicability of the analyticity-ergodicity constraint of Ref. 6.

II. PHYSICAL REALIZATION OF THE SCATTERING MATRIX ENSEMBLES

In the case of the symmetry classes of Altland and Zirnbauer, the role of the chaotic cavity is played by a so-called chaotic Andreev quantum dot,[9,10,19] i.e., a structure formed by a chaotic normal conducting quantum dot contacted to superconductors. In the vicinity of the Fermi level, there are no propagating modes in the superconductors. We consider the situation when the Andreev quantum dot is contacted to normal reservoirs. The number of propagating modes in the contacts to the normal reservoirs (normal contacts for short), including electron-hole degrees of freedom, defines the size of the scattering matrix S. We concentrate on the regime where transport properties can be expressed in terms of the scattering matrix at the Fermi energy: the temperature and the voltages applied to the normal reservoirs are assumed to be much smaller than the energy scale corresponding to the escape rate from the normal region and the gap of the superconductors. (The superconductors are assumed to be grounded.) A sketch of an Andreev quantum dot with two superconducting and one normal contact is shown in Fig. 1. (Charge transport can already take place using one normal contact, due to the Andreev reflection at the superconducting interfaces.[11,12,13,20])

By slightly varying the shape of the Andreev quantum dot, one obtains an ensemble of systems and, therefore, an ensemble of scattering matrices. We discuss below the conditions at which this ensemble can realize the random scattering matrix models discussed in this paper. The only parameters that enter the scattering matrix distribution are the symmetries of S and the properties of the normal contacts. This implies that the conductance of the superconducting contacts should be much larger than of the normal contacts, otherwise transport properties would be sensitive to the ratio of these conductances.[21,22] In addition, Frahm $et\ al$[23] has shown that for the effect of the superconductors on the dynamics in the cavity to be considerable, the Andreev conductances of the superconducting contacts should be much larger than unity.[23] For a random scattering matrix description of transport, it is important that the quasiparticle excitations are gapless. If the excitations were gapped, the normal contacts to the
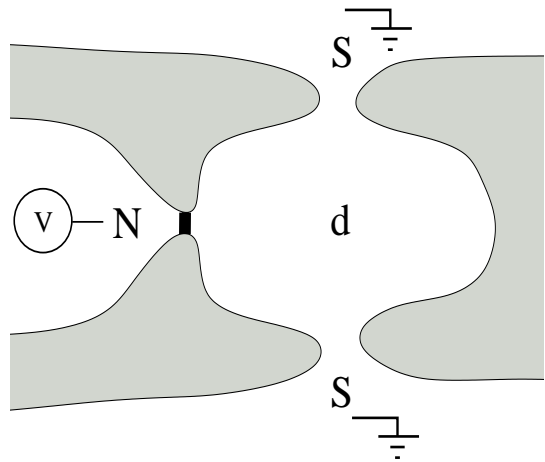

Figure 1: An Andreev quantum dot formed by a normal conducting cavity (d) with two superconducting contacts (S). In the transport state, an infinitesimal voltage V is applied between the grounded superconductors and the normal reservoir (N) contacted at the left opening. The distribution of S derived in this paper considers the effect of a tunnel barrier in the normal contact, indicated by a black rectangle in the figure.

Andreev quantum dot would effectively act as normal-superconductor interfaces,[21,22] directly reflecting incoming quasiparticles, i.e., during transport, the quasiparticles would not explore the chaotic cavity. Gapless chaotic Andreev quantum dots that belong to class C and D can be realized with one superconducting contact already, using a time reversal breaking magnetic field to suppress the proximity gap.[9,23,24,25] For classes CI and DIII, time reversal invariance requires the absence of magnetic fields in the dot. The gap can be suppressed[10,25] by using two superconducting contacts, with a phase difference π. The assumption of a uniformly distributed scattering matrix corresponds to assuming that the coupling of the cavity to the transport channels is ideal, i.e., that the normal contacts are without a tunnel barrier. (The contacts to the superconductors can contain tunnel barriers, as long as they satisfy the aforementioned requirements for their conductances.) In the remaining part of the paper, our task is to generalize this uniform distribution to one that accounts for nonideal normal contacts. It is worthwhile to note here that we do not rely on the specific details of the barriers in the normal contacts, we only use that the scattering matrix of the barriers satisfies the same symmetry requirements as the scattering matrix of the system without the barriers. Our calculation is therefore equally valid for contacts to the normal reservoirs with tunnel barriers that do not mix electrons and holes, and for barriers that mix electrons and holes. The latter situation can occur if there is a region with an induced superconducting gap in the contact to a normal reservoir, that the quasiparticles have to tunnel through to reach the (gapless) cavity region.

III. SCATTERING MATRIX MANIFOLDS

The scattering matrix can be considered as a point of a manifold $\mathcal{M}_X$ in the space of $N \times N$ matrices, where X refers to the symmetry class under consideration. The distribution of the scattering matrix is understood with respect to the invariant measure on $\mathcal{M}_X$. We first state the symmetry properties of $\mathcal{M}_X$ following Ref. 10. We then take a common route,[1,26] and consider $\mathcal{M}_X$ as a Riemannian manifold, to give expressions for the invariant arclength $ds_X^2 = \text{Tr}(dU dU^\dagger)$ and the corresponding measure $d\mu_X(U)$ in an infinitesimal neighborhood of $U \in \mathcal{M}_X$. As usual,[26,27,28] we parametrize this infinitesimal neighborhood with the help of infinitesimal matrices δU_X, with symmetry properties dictated by those of $\mathcal{M}_X$, such that the measure is simply the product of the independent matrix elements of δU_X.

For class D, the manifold $\mathcal{M}_D$ is isomorphic to $\text{SO}(N)$, with $U = \nu \mathcal{O} \nu^\dagger$, $\mathcal{O} \in \text{SO}(N)$, and

$$\nu = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}, \quad \nu^2 = \Sigma_1. \quad (2)$$

It might be worthwhile to note here, that solely from the unitarity of U and the symmetry $U = \Sigma_1 U^* \Sigma_1$ only $\det U = \pm 1$ follows. In Ref. 10, the manifold $\mathcal{M}_D$ was identified through the exponentiation of the Bogoliubov-de Gennes Hamiltonian, which leads to $\det U = 1$ due to the mirror symmetry of the energy levels around zero. (The energies are measured relative to the Fermi level.) The invariant arclength and measure can be written using $\delta U_D = \mathcal{O}^T d\mathcal{O}$ as

$$ds_D^2 = \text{Tr}(\delta U_D \delta U_D^T) = 2\sum_{k<l}(\delta U_D)_{kl}^2,$$

$$d\mu_D(U) \propto \prod_{k<l}(\delta U_D)_{kl}. \quad (3)$$

Note that δU_D is antisymmetric due to the orthogonality of $\mathcal{O}$. It is seen that ds_D^2 [and consequently $d\mu_D(U)$] is invariant under $\delta U_D \to W \delta U_D W^T$, with $W \in \text{O}(N)$. Such a transformation also preserves the antisymmetry of δU_D.

The manifold $\mathcal{M}_{\text{DIII}}$ is spanned by $U = \tilde{U}\tilde{U}^R$, $\tilde{U} \in \mathcal{M}_D$. It is worthwhile to conjugate with

$$V = \frac{1}{\sqrt{2}}\begin{pmatrix} -\sigma_1 & \tau \\ \sigma_1 & \tau \end{pmatrix} \quad (4)$$

and define $\tilde{\mathcal{O}} = V^T \nu^\dagger \tilde{U} \nu V \in \text{SO}(N)$. The matrix V is chosen such that $(\nu V)^\dagger = (\nu V)^R$, from which it follows that $\tilde{U}^R = \nu V \tilde{\mathcal{O}}^R V^T \nu^\dagger$. $\mathcal{M}_{\text{DIII}}$ is therefore isomorphic to the manifold spanned by $\mathcal{O} = \tilde{\mathcal{O}}\tilde{\mathcal{O}}^R$, $\tilde{\mathcal{O}} \in \text{SO}(N)$. Defining $\delta\tilde{\mathcal{O}} = \tilde{\mathcal{O}}^T d\tilde{\mathcal{O}}$, the invariant arclength and mea-

sure can be written in terms of $\delta U_{\text{DIII}} = \delta\tilde{\mathcal{O}} + \delta\tilde{\mathcal{O}}^R$ as

$$ds_{\text{DIII}}^2 = \text{Tr}(\delta U_{\text{DIII}}\, \delta U_{\text{DIII}}^T) = 4\sum_{k<l}(da_{kl}^2 + db_{kl}^2),$$

$$d\mu_{\text{DIII}}(U) \propto \prod_{k<l} da_{kl} db_{kl}, \quad (5)$$

where, in spin grading,

$$\delta U_{\text{DIII}} = \begin{pmatrix} da & db \\ db & -da \end{pmatrix}, \quad da = -da^T, \quad db = -db^T. \quad (6)$$

The parametrization (6) follows from $\delta U_{\text{DIII}} = -\delta U_{\text{DIII}}^T$ and $\delta U_{\text{DIII}} = \delta U_{\text{DIII}}^R$. The arclength ds_{DIII}^2 and the measure $d\mu_{\text{DIII}}(U)$ are invariant under $\delta U_{\text{DIII}} \to W \delta U_{\text{DIII}} W^T$, with $W \in \text{O}(N)$. If W also satisfies $W^R = W^{-1}$, such a transformation preserves the symmetries of δU_{DIII}.

In the case of the classes C and $C\text{I}$ we omit the spin degree of freedom, and we use N to denote the size of the scattering matrices without spin. Electron-hole symmetry is now expressed by the relation[10] $U = \Sigma_2 U^* \Sigma_2$, i.e., U is unitary symplectic. For class C this defines $\mathcal{M}_C = \text{Sp}(N)$. The invariant arclength and measure are

$$ds_C^2 = \text{Tr}(\delta U_C\, \delta U_C^\dagger) =$$

$$2\left(\sum_{q=1}^{3}\sum_l (\delta U_{ll}^{(q)})^2 + 2\sum_{q=0}^{3}\sum_{k<l}(\delta U_{kl}^{(q)})^2\right), \quad (7)$$

$$d\mu_C(U) \propto \left(\prod_{q=1}^{3}\prod_l \delta U_{ll}^{(q)}\right)\prod_{q=0}^{3}\prod_{k<l}\delta U_{kl}^{(q)}. \quad (8)$$

Here, $\delta U_C = U^\dagger dU$, with

$$\delta U_C = \mathbb{1}^{(\text{eh})}\delta U^{(0)} + i\sum_{q=1}^{3}\Sigma_q \delta U^{(q)}, \quad (9)$$

where $\mathbb{1}^{(\text{eh})}$ is the identity matrix in electron-hole space, and $\delta U^{(q)}$ are $N/2 \times N/2$ dimensional real matrices. Due to $(\delta U_C)^\dagger = -\delta U_C$, they satisfy $\delta U^{(0)} = -(\delta U^{(0)})^T$, and for $q > 0$, $\delta U^{(q)} = (\delta U^{(q)})^T$. The arclength and the measure are invariant under $\delta U_C \to W \delta U_C W^\dagger$, with $W \in \text{Sp}(N)$. Such a transformation preserves the symmetries of δU_C as well.

The manifold $\mathcal{M}_{C\text{I}}$ is spanned by $U = \tilde{U}\tilde{U}^T$, with $\tilde{U} \in \text{Sp}(N)$. Defining $\delta\tilde{U} = \tilde{U}^\dagger d\tilde{U}$ and decomposing it according to Eq. (9), we define

$$\delta U_{C\text{I}} = \delta\tilde{U} + (\delta\tilde{U})^T = i\Sigma_1 \delta\tilde{U}^{(1)} + i\Sigma_3 \delta\tilde{U}^{(3)}. \quad (10)$$

The invariant arclength and measure are

$$ds_{C\text{I}}^2 = \text{Tr}(\delta U_{C\text{I}}\delta U_{C\text{I}}^\dagger) =$$

$$2\left(\sum_{q=1,3}\sum_l (\delta\tilde{U}_{ll}^{(q)})^2 + 2\sum_{q=1,3}\sum_{k<l}(\delta\tilde{U}_{kl}^{(q)})^2\right), \quad (11)$$

$$d\mu_{CI}(U) \propto \prod_{q=1,3} \prod_{k \le l} \delta \tilde{U}_{kl}^{(q)}. \qquad (12)$$

The arclength and the measure are invariant under $\delta U_{CI} \to W \delta U_{CI} W^\dagger$ with $W \in \mathrm{Sp}(N) \cap \mathrm{O}(N)$. The symmetry of δU_{CI} is also preserved under such a transformation.

IV. SCATTERING MATRIX DISTRIBUTION

The scattering matrix S at nonideal coupling can be represented as a combination of a random $N \times N$ scattering matrix S_0 at ideal coupling and a fixed $2N \times 2N$ scattering matrix S_c responsible for the direct reflections.[3] The matrix S_0 is assumed to be uniformly distributed with respect to the invariant measure on $\mathcal{M}_X$. The matrix S_c is given by

$$S_c = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}. \qquad (13)$$

Here the dimension of all submatrices is $N \times N$, and all of them carries further structure in electron-hole space, and, for classes D and DIII, also in spin space. The matrix r describes direct reflection from the contact, r' describes reflection back to the cavity from the contact and t and t' are the transmission matrices to and from the cavity, respectively. The scattering matrices S_0 and S_c have the same symmetries.

The total scattering matrix S is given by

$$S = r + t' S_0 (1 - r' S_0)^{-1} t, \qquad (14)$$

and the inverse of the relation is

$$S_0 = (t')^{-1}(S - r)(1 - r^\dagger S)^{-1} t^\dagger. \qquad (15)$$

We derive the distribution of S from the uniform distribution of S_0 following a similar logic to the calculations in Refs. 29,30. The starting point of the reasoning is the relation

$$\delta S = M \delta S_0 M^\dagger, \qquad (16)$$

where $\delta S = S^\dagger dS$, $\delta S_0 = S_0^\dagger dS_0$ and

$$M = (1 - S^\dagger r) \, t^{-1}. \qquad (17)$$

The strategy is to express the arclength in an infinitesimal neighborhood $\mathcal{N}_S$ of S as

$$ds^2(S) = \mathrm{Tr}(dS^\dagger dS) = \mathrm{Tr}(\delta S^\dagger \delta S) = \sum_{ij} g_{ij}(S) dx_i dx_j, \qquad (18)$$

where $\{dx_i\}$ denotes the set of independent matrix elements of δU_X in the parametrization of an infinitesimal neighborhood $\mathcal{N}_{S_0}$ of S_0. [$\mathcal{N}_S$ is the image of $\mathcal{N}_{S_0}$ under the mapping (14).] This way we can relate the measure $d\mu(S_0)$ of $\mathcal{N}_{S_0}$ to the measure $d\mu(S)$ of $\mathcal{N}_S$ as[1,26]

$$d\mu(S) \propto |\det \, g(S)|^{1/2} \prod_j dx_j \propto |\det \, g(S)|^{1/2} d\mu(S_0), \qquad (19)$$

where we used that in $\mathcal{N}_{S_0}$, $d\mu(S_0) \propto \prod_j dx_j$. On the other hand, the probability of $\mathcal{N}_S$ is the same as of $\mathcal{N}_{S_0}$, i.e., $P(S) d\mu(S) = d\mu(S_0)$, which gives $P(S) \propto |\det \, g(S)|^{-1/2}$, the distribution we are after.

Parametrizing $\mathcal{N}_S$ with the help of δU_X and $\mathcal{N}_{S_0}$ using $(\delta U_X)_0$, the relation in Eq. (16) can be written as

$$\delta U_X = M'(\delta U_X)_0 M'^\dagger, \qquad (20)$$

where

$$
\begin{aligned}
M' &= \nu^\dagger M \nu && \text{for class } D, && (21\text{a}) \\
M' &= V^T \nu^\dagger \tilde{U}^R M (\tilde{U}_0^R)^\dagger \nu V && \text{for class } D\text{III}, && (21\text{b}) \\
M' &= \tilde{U}^T M \tilde{U}_0^* && \text{for class } C\text{I}, && (21\text{c})
\end{aligned}
$$

and $M' = M$ for class C. Here the matrices $\tilde{U}, \tilde{U}_0$ are used to express S and S_0 for class CI and DIII according to Sec. III, i.e., $S = \tilde{U}\tilde{U}^y$, $S_0 = \tilde{U}_0 \tilde{U}_0^y$ where $y = T$ and $\tilde{U}, \tilde{U}_0 \in \mathcal{M}_C$ for class CI, and $y = R$ and $\tilde{U}, \tilde{U}_0 \in \mathcal{M}_D$ for class DIII. The matrix M' satisfies

$$
\begin{aligned}
M' &= M'^* && \text{for class } D, && (22\text{a}) \\
M' &= M'^* = \tau M' \tau^T, && \text{for class } D\text{III}, && (22\text{b}) \\
M' &= \Sigma_2 M'^* \Sigma_2 && \text{for class } C, && (22\text{c}) \\
M' &= M'^* = \Sigma_2 M' \Sigma_2 && \text{for class } C\text{I}. && (22\text{d})
\end{aligned}
$$

The reality of the matrix elements of M' for class D and DIII follows from the fact that the set of matrices satisfying $A = \Sigma_1 A^* \Sigma_1$ is closed under matrix addition multiplication and inversion, and that the combination $\nu^\dagger A \nu$ is real. We show the proof of $M' = \tau M' \tau^T$ for class DIII. Because of $\nu V = \tau(\nu V)^* \tau^T$, it is enough to show that $\hat{M} = \tau \hat{M}^* \tau^T$, where

$$\hat{M} = \tilde{U}^R M (\tilde{U}_0^R)^\dagger = \tilde{U}^\dagger S(1 - S^\dagger r) t^{-1} S_0^\dagger \tilde{U}_0. \qquad (23)$$

It is easy to see that

$$\tau \hat{M}^* \tau^T = \tilde{U}^R \, \tau S^*(1 - S^T r^*)(t^*)^{-1} \tau^T S_0^R \, (\tilde{U}_0^R)^\dagger. \qquad (24)$$

Using Eq. (15) and the self duality of S and S_c we find

$$S_0^R = \tau t^*(1 - S^T r^*)^{-1} \tau^T S M. \qquad (25)$$

Substituting in (24), and using again the self duality of S leads to the desired result. The reality of the matrix elements of M' for class CI can be proven following analogous steps. The relation $M' = \Sigma_2 M'^* \Sigma_2$ for class C and CI follows from the closedness of the set of matrices satisfying $A = \Sigma_2 A^* \Sigma_2$ under matrix addition multiplication and inversion.

Following from properties (22), the matrix M' has a singular value decomposition

$$M' = WDW', \tag{26}$$

where

$$D = \mathrm{diag}(d_k), \qquad k = 1 \ldots N \qquad\qquad \text{for class } D, \tag{27a}$$

$$D = \mathrm{diag}(d_k)\,\mathbb{1}^{(\mathrm{sp})}, \qquad k = 1 \ldots \frac{N}{2} \qquad\qquad \text{for class } D\mathrm{III}, \tag{27b}$$

$$D = \mathrm{diag}(d_k)\,\mathbb{1}^{(\mathrm{eh})}, \qquad k = 1 \ldots \frac{N}{2} \qquad\qquad \text{for class } C, \tag{27c}$$

$$D = \mathrm{diag}(d_k)\,\mathbb{1}^{(\mathrm{eh})}, \qquad k = 1 \ldots \frac{N}{2} \qquad\qquad \text{for class } C\mathrm{I}, \tag{27d}$$

with $\mathbb{1}^{(\mathrm{sp})}$ being the identity matrix in spin space, and

$$W, W' \in \mathrm{O}(N) \qquad\qquad\qquad \text{for class } D, \tag{28a}$$

$$W = (W^R)^{-1}, W' = (W'^R)^{-1} \in \mathrm{O}(N) \qquad\qquad \text{for class } D\mathrm{III}, \tag{28b}$$

$$W, W' \in \mathrm{Sp}(N) \qquad\qquad\qquad \text{for class } C, \tag{28c}$$

$$W, W' \in \mathrm{Sp}(N) \cap \mathrm{O}(N) \qquad\qquad \text{for class } C\mathrm{I}. \tag{28d}$$

Using the decomposition (26), the invariant arclength reads

$$ds^2(S) = \mathrm{Tr}(\delta U_X \delta U_X^\dagger) = \mathrm{Tr}\left\{[D(\delta U_X')_0 D][D(\delta U_X')_0 D]^\dagger\right\}, \tag{29}$$

where we used the parametrization $(\delta U_X)_0 = W'^\dagger(\delta U_X')_0 W'$. From the properties of W' in Eq. (28) it follows that the matrix $(\delta U_X')_0$ has the same symmetries as $(\delta U_X)_0$. It is easily read off that

$$\sqrt{\det\, g(S)} \propto \prod_{k<l}^N d_k d_l = \prod_k^N d_k^{N-1} = |\det M|^{N-1} \qquad\qquad \text{for class } D, \tag{30a}$$

$$\sqrt{\det\, g(S)} \propto \prod_{k<l}^{N/2} (d_k d_l)^2 = \prod_k^{N/2} d_k^{N-2} = |\det M|^{\frac{N}{2}-1} \qquad\qquad \text{for class } D\mathrm{III}, \tag{30b}$$

$$\sqrt{\det\, g(S)} \propto \prod_{k<l}^{N/2} (d_k d_l)^4 \prod_j^{N/2} d_j^6 = \prod_k^{N/2} d_k^{2N+2} = |\det M|^{N+1} \qquad\qquad \text{for class } C, \tag{30c}$$

$$\sqrt{\det\, g(S)} \propto \prod_{k\leq l}^{N/2} (d_k d_l)^2 = \prod_k^{N/2} d_k^{N+2} = |\det M|^{\frac{N}{2}+1} \qquad\qquad \text{for class } C\mathrm{I}. \tag{30d}$$

The distribution of S is therefore given by

$$P(S) \propto |\det(1 - r^\dagger S)|^{-(N/t+\sigma)}, \tag{31}$$

where $t = 1$ in the absence of time reversal invariance and $t = 2$ otherwise, and $\sigma = -1$ in the absence of spin rotation invariance and $\sigma = 1$ otherwise.

V. CONDUCTANCE DISTRIBUTION FOR AN ANDREEV QUANTUM DOT IN A MAGNETIC FIELD.

To illustrate the use of our result, we calculate the conductance distribution for a chaotic Andreev quantum dot in a magnetic field. We assume that the spin-orbit scattering is negligible, i.e., the system belongs to symmetry class C. For simplicity, we consider $N = 2$, which is the minimal dimension of S due to the electron-hole structure. This corresponds to the case that the quantum dot is connected to a normal reservoir via a single mode point contact. A sketch of the system is shown in the inset of Fig. 2. The point contact is assumed to contain a tunnel barrier of transparency Γ. The barrier alone does not mix electrons and holes, therefore its reflection matrix is diagonal in electron-hole space,

$$r = \sqrt{1-\Gamma}\begin{pmatrix} e^{i\xi} & 0 \\ 0 & e^{-i\xi} \end{pmatrix} = \sqrt{1-\Gamma}\,\exp(i\xi\Sigma_3). \tag{32}$$

Here ξ is the phase an electron acquires upon reflection from the barrier. The total scattering matrix S is distributed according to $P(S)$ in the group $\text{Sp}(2) \equiv \text{SU}(2)$. The conductance in units of $4e^2/h$ is given by[11,12,13]

$$G(S) = |S_{\text{he}}|^2. \qquad (33)$$

Writing the total scattering matrix as $S = \exp(i\xi\Sigma_3)\, U$, $U \in \text{SU}(2)$, and using that $|S_{\text{he}}|^2 = |U_{\text{he}}|^2$ and $d\mu_C(S) = d\mu_C(U)$, the conductance distribution is given by

$$P(G) = \int_{\text{SU}(2)} \frac{\delta(G - |U_{\text{he}}|^2)}{|\det(1 - \sqrt{1-\Gamma}U)|^3} d\mu_C(U). \qquad (34)$$

Using the Euler angle parameterization for $\text{SU}(2)$,

$$U = \begin{pmatrix} e^{-i(\phi+\psi)/2}\cos(\theta/2) & -e^{i(\psi-\phi)/2}\sin(\theta/2) \\ e^{i(\phi-\psi)/2}\sin(\theta/2) & e^{i(\phi+\psi)/2}\cos(\theta/2) \end{pmatrix},$$

$$(\phi,\psi,\theta) \in [0,2\pi] \times [0,4\pi] \times [0,\pi] \equiv \mathcal{D}, \qquad (35)$$

the measure is $d\mu_C \propto \sin(\theta)$, and $|U_{\text{he}}|^2 = \sin^2(\theta/2)$. The integral

$$P(G) = \frac{\Gamma^3}{16\pi^2} \int_{\mathcal{D}} d\theta d\phi d\psi F_\Gamma(\theta,\phi,\psi)$$

$$F_\Gamma(\theta,\phi,\psi) = \frac{\sin(\theta)\delta[G - \sin^2(\theta/2)]}{[2 - \Gamma - 2\sqrt{1-\Gamma}\cos\frac{\phi+\psi}{2}\cos\frac{\theta}{2}]^3} \qquad (36)$$

can be evaluated in closed form, resulting in

$$P(G) = \Gamma^3 \frac{\Gamma^2 + 2(G-3)(\Gamma-1)}{[\Gamma^2 - 4G(\Gamma-1)]^{5/2}} \qquad (37)$$

for $0 \leq G \leq 1$, and 0 otherwise. In Fig. 2 we show $P(G)$ for different values of the barrier transparency. It is seen that the uniform distribution $P(G) = 1$ corresponding to ideal coupling ($\Gamma = 1$) is gradually transformed into a distribution that is peaked at $G = 0$ as the transparency decreases. The first two moments of the conductance are given by

$$\langle G \rangle = \frac{\Gamma^2}{2}, \quad \langle G^2 \rangle = \frac{\Gamma^3}{3}. \qquad (38)$$

VI. CONCLUSION

In conclusion, we have calculated the distribution $P(S)$ of the scattering matrix at the Fermi energy for chaotic Andreev quantum dots in the nonstandard symmetry classes of Altland and Zirnbauer. Our result, which allows for arbitrary coupling to the transport channels, is based on the assumption that the scattering matrix is
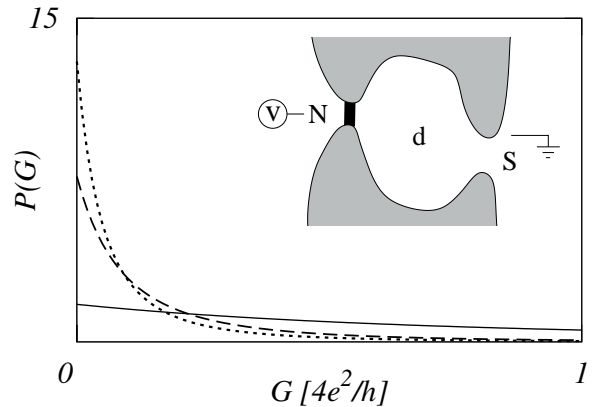

Figure 2: Conductance distribution for a single mode chaotic Andreev quantum dot in a magnetic field for different values of the barrier transparency Γ. A sketch of the system is shown in the inset. The barrier is indicated by a black rectangle. The solid, dashed, and dotted curves correspond to $\Gamma = 0.9$, $\Gamma = 0.6$, and $\Gamma = 0.5$, respectively.

uniformly distributed in $\mathcal{M}_X$ for the case of ideal coupling, i.e., in the absence of direct reflections from the openings of the Andreev quantum dot.

Apart from the symmetry class dependent exponent, our result $P(S)$ has a similar structure to the Poisson kernel distribution $P_\beta(S)$ corresponding to Dyson's standard symmetry classes. As a closing remark, we would like to emphasize an aspect in which $P(S)$ and $P_\beta(S)$ are different. $P_\beta(S)$ can be obtained[6] as the distribution of unitary matrices (with the symmetry corresponding to β) that maximizes the information entropy, subject to the constraint $\langle S^p \rangle = \langle S \rangle^p \equiv r^p$, where r is a subunitary matrix. This analyticity-ergodicity constraint follows[6] from the requirement that the scattering matrix has poles only in the lower half of the complex-energy plane (analyticity), and the assumption that spectral averages equal ensemble averages (ergodicity). Given the similar form of $P(S)$ and $P_\beta(S)$, one might wonder whether $P(S)$ can be obtained by the same maximization procedure. As we show below, the answer is negative. In the presence of superconductivity, spectral average can mean two types of averages. First, it can refer to averaging scattering matrices over an interval of excitation energies ε. Since electron hole-symmetry relates scattering matrices at ε and $-\varepsilon$, it results in the additional constraint $S = \Sigma_1 S^* \Sigma_1$ only at $\varepsilon = 0$. Therefore, such an average would be over scattering matrices with different symmetry than the matrices in the ensemble corresponding to $\varepsilon = 0$, which violates the ergodicity assumption. Second, the spectral average can refer to an average over an interval of Fermi energies of the superconductor, while the excitation energy is kept at $\varepsilon = 0$. However, if $\mathcal{E}$ is a pole of the scattering matrix as the function of the Fermi energy, so is $\mathcal{E}^*$. To see this, one turns to the channel

coupled model used in Ref. 10, in which the poles of the scattering matrix on the complex-Fermi energy plane are eigenvalues of a matrix $(\mathcal{H} - iWW^\dagger)\Sigma_3$, where $\mathcal{H}$ models the Bogoliubov de Gennes Hamiltonian (at a fixed Fermi energy) and W is a coupling matrix. The conjugation relation between the poles is the consequence of $\Sigma_1(\mathcal{H} - iWW^\dagger)^*\Sigma_1 = -(\mathcal{H} - iWW^\dagger)$. This precludes the use of the analyticity properties as in Ref. 6. We thus find that the analyticity-ergodicity constraint is not applicable to the ensembles studied in this paper. This is already signaled in the case of ideal coupling. For example for class D, one has[10] $\langle S \rangle = 0$, but $\langle \mathrm{Tr} S^2 \rangle = 1$. For the more general, nonideal coupling case described by $P(S)$, one also finds that $r \neq \langle S \rangle$, in contrast to the case of the Poisson kernel $P_\beta(S)$. This can be illustrated on the example in Sec. V, where, for $\xi = 0$, $\langle \mathrm{Tr} S \rangle = \sqrt{1 - \Gamma}(\Gamma + 2)$, as opposed to $\mathrm{Tr}\, r = 2\sqrt{1 - \Gamma}$.

[1] C. W. J. Beenakker, Rev. Mod. Phys. **69**, 731 (1997).

[2] F. Dyson, J. Math. Phys. **3**, 140 (1962).

[3] P. W. Brouwer, Phys. Rev. B **51**, 16878 (1995).

[4] L. K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in the Classical Domains* (American Mathematical Society, 1963).

[5] T. J. Krieger, Annals of Physics **42**, 375 (1967).

[6] P. A. Mello, P. Pereyra, and T. Seligman, Ann. Phys. **161**, 254 (1985).

[7] E. Doron and U. Smilansky, Nucl. Phys. A **545**, 455 (1992).

[8] H. U. Baranger and P. A. Mello, Europhys. Lett. **33**, 465 (1996).

[9] A. Altland and M. R. Zirnbauer, Phys. Rev. Lett. **76**, 3420 (1996).

[10] A. Altland and M. R. Zirnbauer, Phys. Rev. B **55**, 1142 (1997).

[11] Y. Takane and H. Ebisawa, J. Phys. Soc. Jpn. **60**, 3130 (1991).

[12] C. J. Lambert, J. Phys. C. **3**, 6579 (1991).

[13] C. W. J. Beenakker, Phys. Rev. B **46**, 12841 (1992).

[14] M. Büttiker, Phys. Rev. B **33**, 3020 (1986).

[15] M. Büttiker, IBM J. Res. Dev. **32**, 63 (1988).

[16] C. M. Marcus, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, Phys. Rev. B **48**, 2460 (1993).

[17] H. U. Baranger and P. A. Mello, Phys. Rev. B **51**, 4703 (1995).

[18] P. W. Brouwer and C. W. J. Beenakker, Phys. Rev. B **55**, 4695 (1997).

[19] C. W. J. Beenakker, Lect. Notes Phys. **667**, 131 (2005).

[20] G. E. Blonder, M. Tinkham, and T. M. Klapwijk, Phys. Rev. B **25**, 4515 (1982).

[21] A. A. Clerk, P. W. Brouwer, and V. Ambegaokar, Phys. Rev. B **62**, 10226 (2000).

[22] P. Samuelsson and M. Büttiker, Phys. Rev. Lett. **89**, 046601 (2002).

[23] K. M. Frahm, P. W. Brouwer, J. A. Melsen, and C. W. J. Beenakker, Phys. Rev. Lett. **76**, 2981 (1996).

[24] J. A. Melsen, P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Europhys. Lett. **35**, 7 (1996).

[25] J. A. Melsen, P. W. Brouwer, K. M. Frahm, and C. W. J. Beenakker, Physica Scripta **T69**, 223 (1997).

[26] C. A. Tracy and H. Widom, Lect. Notes Phys. pp. 103–130 (1994).

[27] M. L. Mehta, *Random Matrices* (Elsevier Ltd., 2004), 3rd ed.

[28] T. Guhr, A. Muller-Groeling, and H. A. Weidenmuller, Phys. Rep. **299**, 189 (1998).

[29] W. A. Friedman and P. A. Mello, Ann. Phys. **161**, 276 (1985).

[30] V. A. Gopar, M. Martínez-Mares, and R. A. Méndez-Sánchez, J. Phys. A: Math. Theor. **41**, 015103 (2008).